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# Skorohod and Stratonovich line integrals in the plane

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In this paper stochastic line and $J$ integrals in the plane in the Skorohod and Stratonovich sense are studied. Different kinds of traces are introduced to deduce the relationships between the Skorohod and Stratonovich integrals. A Green formula is obtained in both cases.

two-parameter processes * anticipative calculus * line integrals

## 1. Introduction

In recent years an anticipative stochastic calculus has been developed by different authors. We refer to the papers by Nualart and Pardoux (1988) and Nualart (1988) for an exposition of results. While in these articles we see that the Skorohod integral is studied, the Stratonovich integral receives a new impulse. Moreover, the multiple Skorohod integral has been studied by Nualart and Zakai (1988) and Jolis and Sanz (1990), and a generalized Itô formula in the plane has been established by the latter authors.

On the other hand, Cairoli and Walsh (1975) introduced, among many other concepts, that of stochastic line integrals in the plane, and proved a Green formula.

In this work the line stochastic integrals in the plane in the Skorohod and Stratonovich sense are defined, and a Green formula is obtained in both cases.

The critical problem with the Skorohod line integral is that of choosing a 'good definition'. Our choice is to define the line integral as the Skorohod (surface) integral of a process which is a kind of projection (vertical or horizontal) of the original process on the curve. Indeed, it is well known that the ordinary stochastic line integrals with respect to a two-parameter Wiener process can be considered as surface integrals of weakly adapted processes.

A Skorohod integral is also defined with respect to the 'measure' $J$. In this case, the definition proposed is also given as a double integral. The classic $J$-integral can also be obtained by this procedure. After defining the mixed integrals, a Green formula is obtained. We should point out that no complementary term appears in this formula, unlike what happens with the Itô formula. But as we see in the proof, the Green formula is a sort of Fubini theorem.

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The Stratonovich surface, line and $J$ integrals are defined as limits in probability which seem to be the most natural definitions. The introduction of different *traces* allows us to obtain relationships with the Skorohod integrals, and to deduce a Green formula. We should point out that strong conditions of regularity are needed and the introduction of the limits $DX^{++}$, $DX^{+-}$, $DX^{-+}$ and $DX^{--}$, and unidimensional limits $DX^{+}$ and $DX^{-}$ required to write the traces, make the notations a little heavy. On the other hand, one of characteristics which makes the ordinary Stratonovich Calculus very interesting is that it reproduces the rules of ordinary differential calculus. However the Green–Stratonovich formula obtained in this paper is not comparable with that of ordinary calculus, and moreover it still has one more term than the Green–Skorohod formula.

The paper is organized as follows: In Section 2, the notations are introduced and some results of Malliavin Calculus are recalled. Section 3 deals with the Skorohod case and is divided into three subsections devoted to the line integral, the $J$-integral and the Green formula. In the fourth section, the Stratonovich case is studied and is divided into four parts: the surface integral, the line integral, the $J$-integral and the Green formula.

## 2. Notations

Denote by $T$ the unity square $[0,1]^2$ and consider the usual partial ordering $(s,t) \leqslant (s',t')$ if $s \leqslant s'$ and $t \leqslant t'$; we will write $(s,t) < (s',t')$ if $s < s'$ and $t < t'$. We also write $(s,t) \curlywedge (s',t')$ if $s \leqslant s'$ and $t \geqslant t'$. For $z, z' \in T$, $z < z'$, $]z, z']$ will be the set $\{\xi \in T: z < \xi \leqslant z'\}$, and $[z, z']$ is defined similarly. Further the rectangle $[0, z]$ is designated by $R_z$.

The probability space $(\Omega, \mathscr{F}, P)$ will be the canonical space associated to the two-parameter Wiener process $\{W_z, z \in T\}$, that is, $\Omega$ is the space of all continuous functions $\omega : T \to \mathbb{R}$ which vanish on the axes, $P$ is the two-parameter Wiener measure and $\mathscr{F}$ is the completion of the Borel $\sigma$-field of $\Omega$ with respect to $P$.

Let $F \in L^2(\Omega)$ be any square integrable functional. The Wiener-chaos expansion of $F$ is:

$$F = \sum_{m=0}^{\infty} I_m(f_m)$$

where $f_m \in L^2(T^m)$ is symmetric and $I_m(f_m)$ is the multiple Wiener–Itô integral. The *k-iterated Malliavin derivative* of $F$ is defined as the stochastic process with $k$ dimensional parameter $D^kF \in L^2(T^k \times \Omega)$ given by

$$(D^kF)(z_1, \ldots, z_k) = \sum_{m=k}^{\infty} \frac{m!}{(m-k)!} I_{m-k}(f_m(z_1, \ldots, z_k)),$$

provided that this sum converges in $L^2(T^k \times \Omega)$. We also write $D^k_{z_1 \ldots z_k}F$ for $(D^kF)(z_1, \ldots, z_k)$. The domain of $D^k$ is denoted by $\mathbb{D}^{k,2}$.

Let $X=\{X(z_1,\ldots,z_k),(z_1,\ldots,z_k)\in T^k\}\in L^2(T^k\times\Omega)$ be a $k$-dimensional stochastic process. We can consider the expansion

$$X(z_1,\ldots,z_k)=\sum_{m=0}^{\infty} I_m(f_m(z_1,\ldots,z_k,\cdot)) \tag{2.1}$$

where $f_m\in L^2(T^{m+k})$, and for any $(z_1,\ldots,z_k)\in T^k$, $f_m(z_1,\ldots,z_k,\cdot)$ is a symmetric function in $L^2(T^m)$. Let $\tilde{f}_m$ be the symmetrization of $f_m$ with respect to all its $m+k$ variables. The *multiple Skorohod integral* of $X$ is defined as

$$\delta^k X=\sum_{m=0}^{\infty} I_{m+k}(\tilde{f}_m)$$

provided that this series converges in $L^2(\Omega)$. The domain of $\delta^k$ is denoted by Dom $\delta^k$.

The multiple Skorohod integral $\delta^k$ coincides with the dual operator of $D^k$ in the following sense: For $F\in\mathbb{D}^{k,2}$ and $X\in\text{Dom }\delta^k$,

$$\langle D^kF, X\rangle_{L^2(T^k\times\Omega)}=\langle F,\delta^kX\rangle_{L^2(\Omega)}.$$

We will denote by $\mathbb{L}^{n,2}_{(k)}$ the class of processes $X\in L^2(T^k\times\Omega)$ such that $X(z_1,\ldots,z_k)\in\mathbb{D}^{nk,2}$ for almost all (Lebesgue) $(z_1,\ldots,z_k)\in T^k$, and there exists a measurable version of

$$\{D^{nk}_{z_1\ldots z_{nk}}X_{z'_1\ldots z'_k},(z_1,\ldots,z_{nk},z'_1,\ldots,z'_k)\in T^{(n+1)k}\}$$

such that

$$E\int_{T^{(n+1)k}}(D^{nk}_{z_1\ldots z_{nk}}X_{z'_1\ldots z'_k})^2\,dz_1\cdots dz_{nk}\,dz'_1\cdots dz'_k<\infty.$$

Then $\mathbb{L}^{1,2}_{(k)}\subset\text{Dom }\delta^k$. When $k=1$, the subindex $k$ will be omitted. If the representation of $X$ is given by (2.1), then $X\in\mathbb{L}^{1,2}_{(k)}$ if and only if

$$\sum_{m=k}^{\infty}\frac{m!}{(m-k)!}\,m!\|f_m\|^2_{L^2(T^{m+k})}<\infty. \tag{2.2}$$

We endow $\mathbb{L}^{1,2}_{(k)}$ with the norm

$$\|X\|_{\mathbb{L}^{1,2}_{(k)}}=\|X\|_{L^2(T^k\times\Omega)}+\sum_{j=1}^{k}\|D^jX\|_{L^2(T^{k+j}\times\Omega)},$$

and then $\mathbb{L}^{1,2}_{(k)}$ is a Banach space. If $X_m, X\in\mathbb{L}^{1,2}_{(k)}$ and $\lim_m X_m=X$ in $\mathbb{L}^{1,2}_{(k)}$, then $\lim_n\delta^k(X_n)=\delta^k(X)$ in $L^2(\Omega)$.

For the proof of those properties, and more results, we refer to Nualart and Pardoux (1988), Nualart and Zakai (1988) and Jolis and Sanz (1990).

## 3. The Skorohod case

### *3.1. Line Skorohod integrals*

First, the definitions about the different types of curves in $T$ introduced by Cairoli and Walsh (1975) are recalled. Let $\Gamma$ be a curve in $T$ given by the parametric representation $\{z \in T: z = \gamma(\sigma), \sigma \in [a, b]\}$, where $\gamma: [a, b] \to T$ is a continuous function. $\hat{\Gamma}$ will denote the curve of the opposite orientation, which has the representation $\{z \in T: z = \hat{\gamma}(\sigma) = \gamma(b + a - \sigma), \sigma \in [a, b]\}$. It is said that $\Gamma$ is of type I if it is an increasing path; of type II if $\sigma \leqslant \sigma'$ implies $\gamma(\sigma) \curlywedge \gamma(\sigma')$; and of type I′ (resp. II′) if $\hat{\Gamma}$ is of type I (resp. II). It is said that $\Gamma$ is of pure type if it is of type I, II, I′ or II′.

Let $\Gamma$ be a curve of pure type, given by $\gamma: [a, b] \to T$, and let $H_z$ (resp. $V_z$) be the horizontal (resp. vertical) line segment connecting $z$ and the $y$-axis (resp. $x$-axis). $D_1$ denotes the closed area bounded by $V_{\gamma(a)}$, $V_{\gamma(b)}$, $\Gamma$ and the axis. Define

$$\tau_1(s) = \begin{cases} \inf\{t': (s, t') \in \Gamma\} & \text{if } (s, 0) \in D_1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\Gamma_1(z) = (s, \tau_1(s))$, that is, if $z \in D_1$, $\Gamma_1(z)$ is the vertical projection of the point $z$ on $\Gamma$.

Given a process on the curve, $X = \{X_z, z \in \Gamma\}$, a process on $T$ is defined by

$$X^{\Gamma_1}(z) = X(\Gamma_1(z))\mathbf{1}_{D_1}(z).$$

**Definition 3.1.** Let $\Gamma$ be a curve of pure type and $X = \{X_z, z \in \Gamma\}$ such that the associated process on $T$, $X^{\Gamma_1}$, belongs to Dom $\delta$. Then it is said that $X$ is *1-Skorohod integrable on* $\Gamma$ and the 1-Skorohod line integral of $X$ is defined by

$$\delta^{\Gamma_1}(X) = \delta(X^{\Gamma_1})$$

if $\Gamma$ is of type I or II, and

$$\delta^{\Gamma_1}(X) = -\delta(X^{\Gamma_1})$$

if $\Gamma$ is of type I′ or II′.

The set of 1-Skorohod integrable on $\Gamma$ processes will be denoted by Dom $\delta^{\Gamma_1}$.

By analogy, $D_2$ is defined as the closed area bounded by $H_{\gamma(a)}$, $H_{\gamma(b)}$, $\Gamma$ and the axes. The horizontal projected process $X^{\Gamma_2}$ and the 2-Skorohod integral on $\Gamma$, $\delta^{\Gamma_2}X$, are introduced in a similar way. If $X \in \text{Dom}\,\delta^{\Gamma_1} \cap \text{Dom}\,\delta^{\Gamma_2}$ it is said that $X$ is *Skorohod integrable on* $\Gamma$ and we write

$$\delta^{\Gamma}(X) = \delta^{\Gamma_1}(X) + \delta^{\Gamma_2}(X).$$

**Proposition 3.2.** *Let $z \in \Gamma$ and denote by $\Gamma'$ the portion of $\Gamma$ between $\gamma(a)$ and $z$, and by $\Gamma''$ the portion between $z$ and $\gamma(b)$. Then* Dom $\delta^{\Gamma_1} \supset$ Dom $\delta^{\Gamma'_1} \cap$ Dom $\delta^{\Gamma''_1}$, *and if* $X \in$ Dom $\delta^{\Gamma'_1} \cap$ Dom $\delta^{\Gamma''_1}$,

$$\delta^{\Gamma_1}(X) = \delta^{\Gamma'_1}(X) + \delta^{\Gamma''_1}(X).$$

**Proof.** It is obvious. □

Let $\Gamma$ be a curve of pure type, with initial point $(s_0, t_0)$ and final point $(s_1, t_1)$. Suppose $s_0 < s_1$ (if not, the obvious changes should be made). We introduce a measure $\mu_{\Gamma_1}$ on $([s_0, s_1], \mathscr{B}([s_0, s_1]))$ related with the vertical projection on $\Gamma$. More precisely, for $z, z' \in \Gamma$ denote by $D^1_{zz'}$ the closed area bounded by $V_z$, $V_{z'}$, $\Gamma$ and the axis, and define

$$\mu_{\Gamma_1}((r, r']) = \lambda(D^1_{(r,\tau_1(r)),(r',\tau_1(r'))}),$$

where $\lambda$ is the Lebesgue measure on $T$ and $s_0 \leq r < r' \leq s_1$. Note that $\mu_{\Gamma_1}$ is absolutely continuous with respect to the Lebesgue measure on $[s_0, s_1]$ and $\mathrm{d}\mu_{\Gamma_1}/\mathrm{d}x = \tau_1$.

**Proposition 3.3.** *Let $\Gamma$ be a curve of pure type and $X = \{X_z, z \in \Gamma\}$ be a process such that $X^{\Gamma_1} \in L^2(T \times \Omega)$ and is* 1*-adapted (i.e., $X^{\Gamma_1}(s, t)$ is $\mathscr{F}_{(s,1)}$ measurable, where $\mathscr{F}_z$ is the $\sigma$-field generated by $\{W_\xi, \xi \in R_z\}$ and the nulls sets of $\mathscr{F}$). Then $X \in \mathrm{Dom}\,\delta_{\Gamma_1}$ and*

$$\delta^{\Gamma_1}(X) = \int_\Gamma X\, \partial_1 W,$$

*where this last integral is the line integral defined by Cairoli and Walsh* (1975).

**Proof.** $X^{\Gamma_1} \in L^2(T \times \Omega)$ and is 1-adapted. Then $X^{\Gamma_1}$ is integrable in the Hajek–Wong sense (see Hajek and Wong, 1983; Nualart and Zakai, 1988; take the collection $\mathscr{C}$ as the class of rectangles $R_{(s,1)}$, $s \in [0, 1]$, and then $X^{\Gamma_1} \in \mathrm{Dom}\,\delta$, and also

$$X^{\Gamma_1} \circ W = \delta(X^{\Gamma_1}),$$

where $\circ$ denotes the Hajek–Wong integral. But by the construction of $\mathscr{C}$, $X^{\Gamma_1} \circ W$ coincides with the '1-weak integral' (see Cairoli and Walsh, 1975, p. 127) and it is well known that the line integral of Cairoli and Walsh can be obtained as a weak integral (see Walsh, 1976, VI. 6). □

**Notations 3.4.** For $X \in \mathrm{Dom}\,\delta^{\Gamma_1}$ we also write

$$\delta^{\Gamma_1}(X) = \int_\Gamma X\, \partial_1 W.$$

We are now going to study the approximation by Riemann sums of the line integrals. Let $\Gamma$ be a pure type curve with initial point $(s_0, t_0)$ and final point $(s_1, t_1)$, and assume $s_0 < s_1$. Let $X = \{X_z, z \in \Gamma\}$ be a process on $\Gamma$ and let $\{\Pi(n), n \geq 1\}$ be a sequence of partitions of $[s_0, s_1]$ such that $\Pi(n) \subset \Pi(n+1)$ and $\lim_n |\Pi(n)| = 0$. Write $\Pi(n) = \{s_0 = \sigma_1^{(n)} < \cdots < \sigma_{r_n}^{(n)} = s_1\}$, put $\Delta_i^n = (\sigma_i^{(n)}, \sigma_{i+1}^{(n)}]$ and denote by $D_1^{n,i}$ the closed area bounded by $V(\sigma_i^{(n)}, \tau_1(\sigma_i^{(n)}))$, $V(\sigma_{i+1}^{(n)}, \tau_1(\sigma_{i+1}^{(n)}))$, $\Gamma$ and the axis, and define

$$X^n(z) = \sum_i \frac{1}{\mu_{\Gamma_1}(\Delta_i^n)} \left( \int_{\Delta_i^n} X(\sigma, \tau_1(\sigma))\, \mathrm{d}\mu_{\Gamma_1}(\sigma) \right) \cdot \mathbf{1}_{D_1^{n,i}}(z), \quad z \in \Gamma.$$

**Proposition 3.5.** *If $X^{\Gamma_1} \in \mathbb{L}^{1,2}$, then $X^n \in \mathrm{Dom}\,\delta$ and*

$$\delta^{\Gamma_1}(X) = L^2(\Omega) - \lim \delta(X^n).$$

**Proof.** $X^n \in \mathbb{L}^{1,2}$ by Proposition 3.1 of Jolis and Sanz (1990) and

$$X^n(s, \tau(s)) = E'[X/G^n \otimes \mathscr{F}](s),$$

where $G^n$ is the $\sigma$-algebra over $[s_0, s_1]$ generated by $\Pi(n)$, and $E'$ is the expectation with respect to $(1/\mu_{\Gamma_1}([s_0, s_1]))\mu_{\Gamma_1} \otimes P$. (Note that $\mathbf{1}_{D_i^{n,i}}(s, \tau_1(s)) = \mathbf{1}_{\Delta_i^n}(s)$.) Then

$$X^n(s, \tau_1(s)) \xrightarrow[n\to\infty]{} X(s, \tau_1(s)) \quad \text{in } L^2([s_0, s_1] \times \Omega, \mu_{\Gamma_1} \otimes P).$$

By the same argument and using the dominated convergence theorem, the convergence of derivatives is obtained. $\square$

**Remarks 3.6.** (1) Let $\Gamma$ be a curve of pure type given by the parametrization $\gamma : [a, b] \to T$, $\gamma(a) = (s_0, t_0)$ and $\gamma(b) = (s_1, t_1)$, and suppose $s_0 < s_1$. A random variable $F \in L^2(\Omega)$ is said to be a *smooth functional* if $F = f(W_{z_1}, \ldots, W_{z_n})$, where $f$ is a $\mathscr{C}^\infty$ function with bounded derivatives of all orders. Let $\mathscr{S}$ be the class of smooth functionals. For $F \in \mathscr{S}$ the 1-derivative along $\Gamma$ can be defined by

$$(D^{\Gamma_1}F)_s = \int_0^{\tau_1(s)} D_{(s,t)}F \,\mathrm{d}t, \quad s \in [s_0, s_1].$$

Then $D^{\Gamma_1}F \in L^2([s_0, s_1] \times \Omega)$. (It must be noted that $\tau_1(s)$ is non-increasing or non-decreasing and bounded.) As usual, a seminorm in $\mathscr{S}$ is defined by

$$\|F\|_{\Gamma_1} = \|F\|_{L^2(\Omega)} + \|D^{\Gamma_1}F\|_{L^2([s_0,s_1]\times\Omega)},$$

and we denote by $\mathbb{D}^{1,2}_{\Gamma_1}$ the completion of $\mathscr{S}$ with respect to this seminorm. The operators $D^{\Gamma_1}$ and $\delta^{\Gamma_1}$ are in duality, in the sense that if $F \in \mathbb{D}^{1,2}_{\Gamma_1}$ and $X \in \operatorname{Dom} \delta^{\Gamma_1}$,

$$E[\delta^{\Gamma_1}(X)F] = E\left[\int_{s_0}^{s_1} (D^{\Gamma_1}F)_s X(s, \tau_1(s)) \,\mathrm{d}s\right].$$

This formula is proved as follows: If $F \in \mathscr{S}$,

$$\begin{aligned} E[\delta^{\Gamma_1}(X)F] &= E[\delta(X(\Gamma_1(z))\mathbf{1}_{D_1}(z))F] \\ &= E\left[\int_{D_1} X(\Gamma_1(z)) D_z F \,\mathrm{d}z\right] \\ &= E\left[\int_{s_0}^{s_1} \int_0^{\tau_1(s)} X(s, \tau_1(s)) D_{st}F \,\mathrm{d}s \,\mathrm{d}t\right] \end{aligned}$$

because of $X(\Gamma_1(z))\mathbf{1}_D(z) \in L^2(T \times \Omega)$ and $DF \in L^2(T \times \Omega)$, and then the Fubini theorem can be applied. For a general $F \in \mathbb{D}^{1,2}_{\Gamma_1}$, an argument of approximation is used.

(2) We have studied the line integral for a process defined on the curve. We also need the line integrals of processes defined on $T$. But the curve is a set of zero measure (Lebesgue), and then when a process $X \in L^2(T \times \Omega)$ is considered it is always assumed that a version of $X$ is fixed.

(3) We will now study the particular case when $\Gamma$ is a horizontal segment. Suppose $\Gamma = H^{(u,v)}$ and the parametrization is given by $\gamma(\sigma) = (\sigma, v)$, $\sigma \in [0, u]$, and then $\tau(s) = v$, $\forall s \in [0, u]$. Define $\mathbb{L}^{1,2}_{H(u,v)}$ as the set of processes $X \in L^2([0, u] \times \Omega)$ such that if we decompose

$$X(s, v) = \sum_{m=0}^{\infty} I_m(f_m(s, v)) \quad \text{a.e. } s \in [0, u],$$

then

$$\sum_{m=1}^{\infty} mm! \int_{T^m} \int_0^u f_m^2((s, v), z_1, \ldots, z_m) \, \mathrm{d}s \, \mathrm{d}z_1 \cdots \mathrm{d}z_m < \infty. \tag{3.1}$$

It is easy to see that then $X^{\Gamma} \in \mathbb{L}^{1,2}$, and this implies $X \in \operatorname{Dom} \delta^{H(u,v)}$.

(4) If $X \in \mathbb{L}^{1,2}$, then $\{X_{sv}, s \in [0, 1]\} \in \mathbb{L}^{1,2}_{H(1,v)}$ a.e. $v \in [0, 1]$.

(5) Let $\Gamma$ be a horizontal segment joining the points $(s_0, t_0)$ and $(s_1, t_0)$, $s_0 < s_1$. Put $B_s = (1/\sqrt{t_0}) W s t_0$, $s \in [s_0, s_1]$. Then $B$ is a standard brownian motion. If $\{X_z, z \in \Gamma\}$ is a stochastic process such that $\{X_{s,t_0}, s \in [s_0, s_1]\} \in \operatorname{Dom} \delta_B$ where $\delta_B$ is the ordinary Skorohod integral with respect to $B$, then $X \in \operatorname{Dom} \delta^{\Gamma_1}$ and

$$\delta^{\Gamma_1}(X) = \sqrt{t_0}\, \delta_B(X_{\cdot t_0}).$$

To see this, suppose first

$$X_{st} = I_n^B f(s, t).$$

Then

$$\begin{aligned} I_n^B f(s, t) &= \int_{[0,1]^n} f(x_1, \ldots, x_n, (s, t)) \, \mathrm{d}B_{x_1} \cdots \mathrm{d}B_{x_n} \\ &= t_0^{-n/2} \int_{T_n} g(z_1, \ldots, z_n, (s, t)) \, \mathrm{d}W_{z_1} \cdots \mathrm{d}W_{z_n} = t_0^{-n/2} I_n(g) \end{aligned}$$

where $g$ is the function defined by

$$g((x_1, y_1), \ldots, (x_n, y_n), (s, t)) = f(x_1, \ldots, x_n, (s, t)) \mathbf{1}_{[0,t_0]^n}(y_1, \ldots, y_n).$$

We can prove this last equality considering first a function $f$ of the form

$$f = \mathbf{1}_{[a_1,b_1] \times \cdots \times [a_n,b_n]}$$

($(s, t)$ is omitted).

Then $g$ will be of the form

$$g = \mathbf{1}_{[a_1,b_1] \times [0,t_0] \times \cdots \times [a_n, b_n] \times [0, t_0]},$$

and

$$\begin{aligned} I_n^B(f) &= (B_{b_1} - B_{a_1}) \cdots (B_{b_n} - B_{a_n}) \\ &= t_0^{-n/2} (W_{(b_1,t_0)} - W_{(a_1,t_0)}) \cdots (W_{(b_n,t_0)} - W_{(a_n,t_0)}) \\ &= t_0^{-n/2} I_n(g). \end{aligned}$$

The same is true for functions $f$ which are linear combinations of functions such as those above, and the property follows by density.

Now, we can consider the Skorohod integral of $X$:

$$\delta_B(X_{st_0}) = I^B_{n+1}(f(s, \widetilde{t_0)\mathbf{1}}_{[s_0 s_1]}(s))$$

and

$$\begin{aligned}\delta^{\Gamma_1}(X) &= \delta(X_{s,t_0}\mathbf{1}_{[s_0,s_1]\times[0,t_0]}(s, t))\\ &= \delta(t_0^{-n/2}I_n(g(s, t_0)\mathbf{1}_{[s_0,s_1]\times[0,t_0]}(s, t)))\\ &= t_0^{-n/2}I_{n+1}(f(x_1, \ldots, x_n, (s, t_0))\widetilde{\mathbf{1}_{[0,t_0]^{n+1}}(y_1, \ldots, y_n, t)\mathbf{1}_{[s_0,s_1]}(s))}\\ &= t_0^{-n/2}t_0^{n+1/2}I^B_{n+1}(f(x_1, \ldots, x_n, \widetilde{(s, t_0)})\mathbf{1}_{[s_0, s_1]}(s))\\ &= t_0^{1/2}\delta_B(X_{\cdot t_0}).\end{aligned}$$

In general the reciprocal implication is not true, that means, $X \in \operatorname{Dom} \delta^{\Gamma_1}$ does not imply $X_{\cdot t_0} \in \operatorname{Dom} \delta^B$. This is because $\operatorname{Dom} \delta^{\Gamma_1}$ is a subspace of $L^2(T\times\Omega, \mathscr{B}(T)\otimes \mathscr{F})$, where $\mathscr{F}$ is the completion of the $\sigma$-field generated by the Brownian sheet, and $\operatorname{Dom} \delta^B$ is a subspace of $L^2([s_0, s_1]\times\Omega, \mathscr{B}([s_0, s_1])\otimes\mathscr{F}^B)$, where $\mathscr{F}^B$ is the $\sigma$-field corresponding to Brownian $B$, and the inclusion $\mathscr{F}^B \subset \mathscr{F}$ is strict.

(6) In general the space $\mathbb{L}^{1,2}_{\Gamma_i}$ is the class of processes $X$ defined on $\Gamma$ such that $\{X(s, \tau(s)), s\in[0, 1]\}$ has a property as (3.1) with the obvious changes.

### 3.2. J-Skorohod integrals

Denote by $\Psi$ the indicator function of the set of points $(z, z')\in T^2$ such that $z \curlywedge z'$.

**Definition 3.7.** Let $X \in L^2(T\times\Omega)$. It is said that $X$ is $J$-Skorohod integrable if the process $\{Y(z, z'), (z, z')\in T^2\}$ defined by

$$Y(z, z') = X(z \vee z')\Psi(z, z')$$

belongs to $\operatorname{Dom} \delta^2$, where $(s_1, t_1)\vee(s_2, t_2) = (\max(s_1, s_2), \max(t_1, t_2))$. In this case, it is written

$$\delta^J(X) = \delta^2(Y).$$

The set of $J$-Skorohod-integrable processes is denoted by $\operatorname{Dom} \delta^J$.

**Proposition 3.8.** *Let $X \in L^2(T\times\Omega)$ be an adapted process. Then $X \in \operatorname{Dom} \delta^J$ and*

$$\delta^J(X) = \int_T X\, \mathrm{d}J,$$

*where the last integral is the Cairoli–Walsh integral with respect to $J$ (see Cairoli and Walsh, 1975).*

**Proof.** By Proposition 4.1 of Nualart and Zakai (1988) to prove that the process $Y$ introduced in Definition 3.1 belongs to Dom $\delta^2$, it suffices (given the particular form of this process) to check that

$$E\int_{T^2} Y(z,z')^2\,\mathrm{d}z\,\mathrm{d}z' < \infty,$$

and this is obtained by applying the Fubini theorem:

$$E\int_{T^2} X(z\vee z')^2\Psi(z,z')\,\mathrm{d}z\,\mathrm{d}z' = E\int_T X^2(s',t)s't\,\mathrm{d}s'\,\mathrm{d}t$$

$$\leqslant E\int_T X^2(z)\,\mathrm{d}z<\infty.$$

Note that the condition $E\int_T X^2(s',t)s't\,\mathrm{d}s'\,\mathrm{d}t<\infty$ is exactly the Cairoli–Walsh's condition for $J$-integrability.

Also, by the same proposition of Nualart and Zakai,

$$\delta^2(Y)=\int_T X_z\,\mathrm{d}J_z$$

the last integral is in the Cairoli–Walsh sense.

To finish the proof, we have only to show that

$$\int\int_{T^2} X(z\vee z')\Psi(z,z')\,\mathrm{d}W_z\,\mathrm{d}W_{z'} = \int_T X_z\,\mathrm{d}J_z$$

(both according to Cairoli and Walsh). To see this, we can apply the stochastic Fubini theorem (Cairoli and Walsh, 1975, Theorem 2.6), and Green's formula (preceding reference, Theorem 6.1). The proof is as follows:

$$\int_T X\,\mathrm{d}J = \int_{H_{11}}\left(\int_{V_{st}} X\,\partial_2 W\right)\partial W - \int_T\left(\int_{V_{st}} X\,\partial_2 W\right)\mathrm{d}W$$

$$=\int_T\left(\int_T X_{sv}\mathbf{1}_{R_{s1}}(u,v)\,\mathrm{d}W_{u,v}\right)\mathrm{d}W_{st}$$

$$-\int_T\left(\int_T X_{sv}\mathbf{1}_{R_{st}}(u,v)\,\mathrm{d}W_{uv}\right)\mathrm{d}W_{st}$$

$$=\int_T\left(\int_{[0,s]\times[\tau,1]} X(z\vee z')\,\mathrm{d}W_z\right)\mathrm{d}W_{z'}$$

$$=\int_{T^2} X(z\vee z')\Psi(z,z')\,\mathrm{d}W_z\,\mathrm{d}W_{z'}. \qquad \square$$

**Notations 3.9.** For $X \in \operatorname{Dom} \delta^J$, $\delta^J(X) = \int_T X \, \mathrm{d}J$ it is also written.

**Proposition 3.10.** $\mathbb{L}^{2,2} \subset \operatorname{Dom} \delta^J$.

**Proof.** Consider the expansion of $X$:

$$X_z = \sum_{n=0}^{\infty} I_n(f_n(z)).$$

Then the expansion of $Y$ is

$$Y(z, z') = \sum_{n=0}^{\infty} I_n(f_n(z \vee z') \Psi(z, z')).$$

and

$$\sum_{n=0}^{\infty} (n+2)! \|\widetilde{f_n(z \vee z') \Psi(z, z')}\|^2_{L^2(T^{n+2})}$$

$$\leqslant \sum_{n=0}^{\infty} (n+2)! \|f_n(z \vee z') \Psi(z, z')\|^2_{L^2(T^{n+2})}$$

$$\leqslant \sum_{n=0}^{\infty} (n+2)! \int \cdots \int_{T^{n+1}} f_n^2(z_1, \ldots, z_{n+1}) \, \mathrm{d}z_1 \cdots \mathrm{d}z_{n+1},$$

where the last inequality is due to the Fubini Theorem. The proposition follows from $X \in \mathbb{L}^{2,2}$. □

### 3.3. Green formula

**Definition 3.11.** Consider a process $X \in L^2(T \times \Omega)$ such that, fixed $(s, t)$, for almost all $r \in [0, t]$, $X \in \operatorname{Dom} \delta^{H(s,r)_1}$, and $\int_{H(s,r)} X \, \partial_1 W$ is Lebesgue integrable in $r$. It can then be defined a mixed integral

$$\iint_{R_{st}} X \, \partial_1 W \, \mathrm{d}r \overset{\text{def}}{=} \int_0^t \left( \int_{H(x,r)} X \, \partial_1 W \right) \mathrm{d}r.$$

**Lemma 3.12.** *Let $f \in L^2(T \times [0, 1] \times \Omega)$ such that for almost all $r \in [0, 1]$, $f(\cdot, r) \in \operatorname{Dom} \delta$, and $\delta(f(\cdot, r))$ is Lebesgue integrable and $\int_0^1 \delta(f(\cdot, r)) \, \mathrm{d}r \in L^2(\Omega)$. Then $\int_0^1 f(\cdot, r) \, \mathrm{d}r \in \operatorname{Dom} \delta$ and*

$$\delta\left( \int_0^1 f(\cdot, r) \, \mathrm{d}r \right) = \int_0^1 \delta(f(\cdot, r)) \, \mathrm{d}r.$$

**Proof.** It suffices to consider $f(z, r) = I_n(f_n(z, r))$, and use the commutation between multiple Wiener integral and Lebesgue integral (see Lemma 1.1 of Sekigushi and Shiota, 1985). □

**Proposition 3.13.** *Let $X \in \mathbb{L}^{1,2}$. Then for any $(s, t)$ there is the mixed integral and*

$$\int\int_{R_{st}} X\, \partial_1 W\, \mathrm{d}r = \int_T \left( \int_0^t X(u, r) \mathbf{1}_{[v,1]}(r)\, \mathrm{d}r \right) \mathbf{1}_{[0,s]}(u)\, \mathrm{d}W_{uv}.$$

**Proof.** We have seen (Remark 3.6.4) that if $X \in \mathbb{L}^{1,2}$ then $X \in \mathrm{Dom}\, \delta^{H_{sr}}$ for almost all $r$. Also

$$\int_{H_{sr}} X\, \partial_1 W = \int_T X(u, r) \mathbf{1}_{[0,s]\times[0,r]}(u, v)\, \mathrm{d}W_{uv}$$

and

$$\begin{aligned} E\left[ \left( \int_{H_{sr}} X\, \partial_1 W \right)^2 \right] &\le C \int_T E[X(u, r)^2 \mathbf{1}_{[0,s]\times[0,r]}(u, v)]\, \mathrm{d}u\, \mathrm{d}v \\ &\quad + C \int_{T^2} E[(D_z X(u, r))^2] \mathbf{1}_{[0,s]\times[0,r]}(u, v)\, \mathrm{d}v\, \mathrm{d}u\, \mathrm{d}z < \infty. \end{aligned}$$

To finish the proof, Lemma 3.12 is applied. □

**Definition 3.14.** Given two measurable processes $f = \{f(z), z \in T\}$ and $g = \{g(z), z \in T\}$ such that for any $(s, r) \in T$, $f \in \mathrm{Dom}\, \delta^{V(s,r)}$, and $\int_0^1 |g(s, v)|\, \mathrm{d}v < \infty$, a.e., consider the process

$$M_{sr} = M_{s0} + \int_{V_{sr}} f\, \partial_2 W + \int_{V_{sr}} g\, \mathrm{d}v.$$

Then it is said that $M$ has 2-Skorohod stochastic partial derivatives $f$ and $g$ in $T$.
Similarly, the 1-Skorohod stochastic partial derivatives are defined.

**Example 3.15.** By Remark 3.6.5,

$$W_{11} W_{st} = \int_{V_{st}} W_{11}\, \partial_2 W + s \int_0^t \mathrm{d}v$$

and therefore, the process $M_{st} = W_{11} W_{st}$ has 2-Skorohod stochastic partial derivatives in $T$ and they are $f = W_{11}$ and $g = s$.

**Theorem 3.16** (Green formula for rectangles). *Let $f \in \mathbb{L}^{2,2}$, $g \in \mathbb{L}^{1,2}$ and let $X$ be the stochastic process given by*

$$X_{st} = X_{s0} + \int_{V_{st}} f\, \partial_2 W + \int_{V_{st}} g\, \mathrm{d}v.$$

*Then, for any rectangle $A \subset T$,*

$$\int_{\partial A} X\, \partial_1 W = \int_A X\, \mathrm{d}W + \int_A f\, \mathrm{d}J + \iint_A g\, \partial_1 W\, \mathrm{d}t,$$

*where the line integral is taken in the clockwise direction.*

**Proof.** It suffices to consider a rectangle $A = R_z$ because we can decompose a general rectangle $A = (z_1, z_2]$ into $(R_{z_2} - R_{(s_1, t_2)}) - (R_{(s_2, t_1)} - R_{z_1})$ and the line and surface integrals are additive. Also it can be assumed that $z = (1, 1)$.

First the case $g = 0$ is studied, and after this, it is supposed that $f = 0$. Assume $X_{s0} = 0$, and let

$$X_{uv} = \int_{V_{uv}} f\, \partial_2 W = \int_T f(u, y) \mathbf{1}_{R_{uv}}(x, y)\, \mathrm{d}W_{xy}$$

and then $X \in \mathbb{L}^{1,2} \subset \operatorname{Dom} \delta$ and $X_{u1} \in \operatorname{Dom} \delta^{H_{1,1}}$, because $f \in \mathbb{L}^{2,2}$ and the definition of $X$.

Now

$$\begin{aligned}
\int_{\partial T} X\, \partial_1 W &= \int_{H_{11}} X\, \partial_1 W = \int_T X_{u1}\, \mathrm{d}W_{uv} \\
&= \int_T \left( \int_T f(u, y) \mathbf{1}_{[0,u]}(x)\, \mathrm{d}W_{xy} \right) \mathrm{d}W_{uv} \\
&= \int_T \left( \int_T f(u, y) \mathbf{1}_{[0,u]}(x) \mathbf{1}_{[0,v]}(y)\, \mathrm{d}W_{xy} \right) \mathrm{d}W_{uv} \\
&\quad + \int_T \left( \int_T f(u, y) \mathbf{1}_{[0,u]}(x) \mathbf{1}_{[v,1]}(y)\, \mathrm{d}W_{xy} \right) \mathrm{d}W_{uv} \\
&= \int_T X\, \mathrm{d}W \\
&\quad + \int_T \left( \int_T f((u, v) \vee (x, y)) \psi((x, y), (u, v))\, \mathrm{d}W_{xy} \right) \mathrm{d}W_{uv} \\
&= \int_T X\, \mathrm{d}W + \int_T f\, \mathrm{d}J.
\end{aligned}$$

Now suppose that $X_{uv} = \int_0^v g(u, r)\, \mathrm{d}r$. Then it is clear that $X \in \mathbb{L}^{1,2}$, and by Proposition 3.13,

$$\begin{aligned}
\int_{\partial T} X\, \partial_1 W &= \int_T X_{u1}\, \mathrm{d}W_{uv} = \int_T \left( \int_0^1 g(u, r)\, \mathrm{d}r \right) \mathrm{d}W_{uv} \\
&= \int_T \left( \int_0^v g(u, r)\, \mathrm{d}r \right) \mathrm{d}W_{uv} + \int_T \left( \int_v^1 g(u, r)\, \mathrm{d}r \right) \mathrm{d}W_{uv} \\
&= \int_T X\, \mathrm{d}W + \iint_T g\, \partial_1 W\, \mathrm{d}r. \qquad \square
\end{aligned}$$

## 4. The Stratonovich case

### 4.1. The Stratonovich integral in the plane

**Definition 4.1.** Let $X=\{X_z, z\in T\}$ be a two parameter measurable stochastic process such that $\int_T X_z^2\,\mathrm{d}z<\infty$ a.s., and let $\Pi=\{(s_i,t_j), 0=s_0<s_1<\cdots<s_{n+1}=1, 0=t_0<t_1<\cdots<t_{m+1}=1\}$ be a partition of $T$. Put

$$\Delta_{ij}=(s_i,s_{i+1}]\times(t_j,t_{j+1}],\qquad |\Delta_{ij}|=(s_{i+1}-s_i)(t_{j+1}-t_j)$$

and

$$|\Pi|=\max_{i,j}\{|\Delta_{ij}|\}.$$

Write

$$S_\Pi(X)=\sum_{i,j}\left(\frac{1}{|\Delta_{ij}|}\int_{\Delta_{ij}}X_z\,\mathrm{d}z\right)W(\Delta_{ij}).$$

If the limit

$$I^s(X)=\lim_{|\Pi|\downarrow 0}S_\Pi(X)$$

exists in the sense of convergence in probability, it is said that $X$ is Stratonovich integrable and $I^s(X)$ is called the Stratonovich integral of $X$. The set of Stratonovich integrable processes will be denoted by Dom $I^s$. We also write

$$I^s(X)=\int_T X\circ\mathrm{d}W.$$

**Definition 4.2.** It is said that a process $X\in\mathbb{L}^{1,2}$ has a *trace* if there exists the limit in probability

$$TX=\lim_{|\Pi|\downarrow 0}\sum_{i,j}\frac{1}{|\Delta_{ij}|}\int_{(\Delta_{ij})^2}D_{z'}X_z\,\mathrm{d}z\,\mathrm{d}z'.$$

**Remark 4.3.** The obvious charges in the proof of Proposition 1.3 from Solé and Utzet (1990) show that in the Definitions 4.1 and 4.2 it suffices to consider only sequences of partitions $\{\Pi(n), n\geqslant 1\}$ of $T$ such that $\Pi(n)\subset\Pi(n+1)$ and $\lim_n|\Pi(n)|=0$. Such sequences are called *sequences of refinements* of $T$ from now.

**Proposition 4.4.** *Let $X\in\mathbb{L}^{1,2}$. Then $X\in$* Dom $I^s$ *if and only if $X$ has a trace. In this case,*

$$I^s(X)=\delta(X)+TX.$$

**Proof.** The proof follows the same ideas of that of Theorem 1.9 from Solé and Utzet (1990). $\square$

**Notations 4.5.** It is said that a process $X \in \mathbb{L}^{1,2}$ is of class $\mathbb{L}_a$ if there exists a version $\{D_z X_{z'}, (z, z') \in T^2\}$ such that there is a neighbourhood $V \subset T^2$ of the diagonal $\{(z, z), z \in T\}$ such that

(i) For any $z$, if we write

$$V_z^{++} = \{z' \in T\colon z < z' \text{ and } (z, z') \in V\},$$

the application

$$V_z^{++} \to L^2(\Omega), \qquad z' \to D_z X_{z'},$$

is continuous, uniformly in $z$.

(ii) Analogously, we define $V_z^{+-}$, $V_z^{-+}$, $V_z^{--}$, and consider the corresponding applications, which are supplied to be continuous on the domains $V^{\alpha\beta}$ uniformly in $z$.

$$\text{(iii)} \qquad \sup_{(z,z') \in V} E[(D_z X_{z'})^2] < \infty.$$

For a process $X \in \mathbb{L}_a$ we can define the limits

$$DX(s^+, t^+) = L^2(\Omega) - \lim_{\substack{s' \downarrow s \\ t' \downarrow t}} D_{st} X_{s't'}$$

(we also write $(s, t)^{++}$ for $(s^+, t^+)$) and $DX(s^+, t^-)$, $DX(s^-, t^+)$, $DX(s^-, t^-)$. Conditións (i), (ii) and (iii) imply that there is a version $\{DX_{z^{++}}, z \in T\} \in L^2(T \times \Omega)$.

At the end, we write

$$\Delta X_z = \tfrac{1}{4}(DX_{z^{++}} + DX_{z^{-+}} + DX_{z^{+-}} + DX_{z^{--}}).$$

**Lemma 4.6.** *Let $\{\Pi(n), n \geq 1\}$ be a sequence of refinements. For any $(s, t) \in \Delta_{ij}^n$, put*

$$\Delta_{ij}^{n++}(s, t) = (s, s_{i+1}] \times (t, t_{j+1}].$$

*Then*

$$\lim_n \sum_{i,j} \frac{|\Delta_{ij}^{n++}(s, t)|}{|\Delta_{ij}^n|} \mathbf{1}_{\Delta_{ij}^n}(s, t) = \tfrac{1}{4}$$

*weakly in $\sigma(L'(T), L^\infty(T))$. As a consequence, for any bounded stochastic process $\{Y_z, z \in T\}$,*

$$\lim_n E\left[\left\{\int_T \left(\sum_{i,j} \frac{|\Delta_{ij}^{n++}(z)|}{|\Delta_{ij}^n|} \mathbf{1}_{\Delta_{ij}^n}(z) - \tfrac{1}{4}\right) Y_z \, \mathrm{d}z\right\}^2\right] = 0.$$

**Proof.** It suffices to show that for any $A \in \mathscr{B}(T)$,

$$\lim_n \int_A \left(\sum_{i,j} \frac{|\Delta_{ij}^{n++}(z)|}{|\Delta_{ij}^n|} \mathbf{1}_{\Delta_{ij}^n}(z) - \tfrac{1}{4}\right) \mathrm{d}z = 0.$$

By standard arguments, we need only consider the case $A=(a_1,b_1]\times(a_2,b_2]$, and then it is a simple calculus. $\square$

**Theorem 4.7.** *Let $X\in\mathbb{L}_a$. Then $X$ has a trace and*

$$TX=\int_T \Delta X_z\,\mathrm{d}z.$$

**Proof.** Let $\{\Pi(n),\ n\geqslant 1\}$ be a sequence of refinements. To simplify the notations, we will omit the superindex $n$ in $\Delta_{ij}^{n++}$ and $\Delta_{ij}^{n}$.

We have

$$\begin{aligned}
&E\Bigg[\Bigg\{\sum_{i,j}\frac{1}{|\Delta_{ij}|}\int_{(\Delta_{ij})^2} D_zX_{z'}\,\mathrm{d}z\,\mathrm{d}z'\\
&\qquad -\frac{1}{4}\int_T (DX_{z^{++}}+DX_{z^{-+}}+DX_{z^{+-}}+DX_{z^{--}})\,\mathrm{d}z\Bigg\}^2\Bigg]\\
&\leqslant 2^2E\Bigg[\Bigg\{\int_T\Bigg(\sum_{i,j}\frac{1}{|\Delta_{ij}|}\int_{\Delta_{ij}^{++}(z)}\mathbf{1}_{\Delta_{ij}(z)}D_zX_{z'}\,\mathrm{d}z'-\tfrac14 DX_{z^{++}}\Bigg)\mathrm{d}z\Bigg\}^2\Bigg]\\
&\qquad +\text{other three similar terms}\\
&\leqslant 2^3E\Bigg[\Bigg\{\int_T\Bigg(\sum_{i,j}\mathbf{1}_{\Delta_{ij}}(z)\frac{|\Delta_{ij}^{++}(z)|}{|\Delta_{ij}|}\\
&\qquad\qquad\times\Bigg(\frac{1}{|\Delta_{ij}^{++}(z)|}\int_{\Delta_{ij}^{++}(z)}D_zX_{z'}\,\mathrm{d}z'-DX_{z^{++}}\Bigg)\Bigg)\mathrm{d}z\Bigg\}^2\Bigg]\\
&\quad +2^3E\Bigg[\Bigg\{\int_T\Bigg(\sum_{i,j}\frac{|\Delta_{ij}^{++}(z)|}{|\Delta_{ij}|}\mathbf{1}_{\Delta_{ij}}(z)-\tfrac14\Bigg)DX_{z^{++}}\,\mathrm{d}z\Bigg\}^2\Bigg]\\
&\qquad +\text{other similar terms.}
\end{aligned}\tag{4.1}$$

By Jensen's inequality, the first term on the right is bounded by

$$\begin{aligned}
&2^3\int_T\Bigg(\sum_{i,j}\mathbf{1}_{\Delta_{ij}}(z)\frac{|\Delta_{ij}^{++}(z)|^2}{|\Delta_{ij}|^2}\frac{1}{|\Delta_{ij}^{++}(z)|}\\
&\qquad\times\int_{\Delta_{ij}^{++}(z)}E[(D_zX_{z'}-DX_{z^{++}})^2]\,\mathrm{d}z'\Bigg)\mathrm{d}z\\
&\leqslant 2^3\int_T\sum_{i,j}\mathbf{1}_{\Delta_{ij}}(z)\sup_{\xi\in\Delta_{ij}^{++}(z)}E[(D_zX_\xi-DX_{z^{++}})^2]\,\mathrm{d}z,
\end{aligned}$$

which goes to zero when $n\to\infty$, by definition of $DX_{z^{++}}$.

For fixed $k>0$, the second term of (4.1) is bounded by

$$B^n = 2^4 E\left[\left\{\int_T \left(\sum_{i,j} \frac{|\Delta_{ij}^{++}(z)|}{|\Delta_{ij}|} \mathbf{1}_{\Delta_{ij}}(z) - \tfrac{1}{4}\right) DX_{z^{++}} \mathbf{1}_{\{DX_{z^{++}}<k\}}(z)\,\mathrm{d}z\right\}^2\right]$$

$$+ 2^4 E\left[\left\{\int_T \left(\sum_{i,j} \frac{|\Delta_{ij}^{++}(z)|}{|\Delta_{ij}|} \mathbf{1}_{\Delta_{ij}}(z) - \tfrac{1}{4}\right) DX_{z^{++}} \mathbf{1}_{\{DX_{z^{++}}\geqslant k\}}(z)\,\mathrm{d}z\right\}^2\right]$$

$$= B_1^{n,k} + B_2^{n,k}$$

where $n$ denotes the partition. By Lemma 4.6, $\lim_{n\to\infty} B_1^{n,k} = 0\ \forall k$, and by Schwarz's inequality,

$$B_2^{n,k} \leqslant \int_T \left(\sum_{i,j} \frac{|\Delta_{i,j}^{++}(z)|}{|\Delta_i||\Delta_j|} \mathbf{1}_{\Delta_i\times\Delta_j}(z) - \tfrac{1}{4}\right)^2 E[(DX_{z^{++}})^2 \mathbf{1}_{\{DX_{z^{++}}>k\}}]\,\mathrm{d}z$$

$$\leqslant \int_T E[DX_{z^{++}})^2 \mathbf{1}_{\{DX_{z^{++}}>k\}}(z)]\,\mathrm{d}z.$$

If we fix $z$,

$$C^k = E[(DX_{z^{++}})^2 \mathbf{1}_{\{DX_{z^{++}}>k\}}] \xrightarrow[k\to\infty]{} 0,$$

and

$$E[(DX_{z^{++}})^2 \mathbf{1}_{\{DX_{z^{++}}>k\}}] \leqslant E[(DX_{z^{++}})^2]$$

and by dominated convergence theorem,

$$\lim_{k\to\infty} C^k = 0 \quad \text{in } L^1(T).$$

Then $\lim_k(\limsup_n B_2^{n,k}) = 0$ because $B_2^{n,k} < C^k$ and the theorem follows. □

**Proposition 4.8.** *Let* $Y \in \mathbb{L}^{2,2}$ *such that* $\sup_{z\in T} E[Y_z^2] < \infty$, $\sup_{z,z'\in T^2} E[(D_z Y_{z'})^2] < \infty$, *and* $\sup_{z,z',z''\in T^3} E[(D_{z,z'}^2 Y_{z''})^2] < \infty$, *and consider the process*

$$X_z = \int_{R_z} Y_\xi\,\mathrm{d}W_\xi.$$

*Then X has a trace, and*

$$TX = \frac{1}{4}\int_T Y_\xi\,\mathrm{d}\xi + \int_T \left(\int_{R_z} D_z Y_\xi\,\mathrm{d}W_\xi\right)\mathrm{d}z.$$

**Proof.** $D_zX_{z'} = Y_z \mathbf{1}_{R_{z'}}(z) + \int_T D_z Y_\xi \mathbf{1}_{R_{z'}}(\xi)\,\mathrm{d}W_\xi$ (see remark following Proposition 3.3 of Nualart and Pardoux, 1988). If $z', z'' > z$, then

$$\begin{aligned}
&E[(D_zX_{z'} - D_zX_{z''})^2] \\
&= E\left[\left(\int_{R_{z'}} D_zY_\xi\,\mathrm{d}W_\xi - \int_{R_{z''}} D_zY_\xi\,\mathrm{d}W_\xi\right)^2\right] \\
&\leqslant 2E\left[\left(\int_{R_{z'}-R_{z''}} D_zY_\xi\,\mathrm{d}W_\xi\right)^2 + \left(\int_{R_{z''}-R_{z'}} D_zY_\xi\,\mathrm{d}W_\xi\right)^2\right] \\
&\leqslant C\int_{R_{z'}\Delta R_{z''}} (E[(D_zY_\xi)^2])\,\mathrm{d}\xi + C\int_T\int_{R_{z'}\Delta R_{z''}} E[(D_{\eta z}Y_\xi)^2]\,\mathrm{d}\eta\,\mathrm{d}\xi
\end{aligned}$$

($\Delta$: symmetric difference) and the conditions for $X \in \mathbb{L}_a$ follow easily. Now, with an analogous reasoning

$$DX_{z^{++}} = Y_z + \int_{R_z} D_zY_\xi\,\mathrm{d}W_\xi.$$

Similarly,

$$DX_{z^{+-}} = DX_{z^{-+}} = DX_{z^{--}} = \int_{R_z} D_zY_\xi\,\mathrm{d}W_\xi. \qquad \square$$

*4.2. The line Stratonovich integral*

**Definition 4.9.** Let $\Gamma$ be a curve of pure type with inicial point $(s_0, t_0)$ and final point $(s_1, t_1)$; suppose $s_0 \leqslant s_1$, and $X$ be a process on the curve, $\{X_z, z \in \Gamma\}$, such that $X(s, \tau(s))$ is $\mu_{\Gamma_1}$-integrable. Then it is said that $X$ is 1-*Stratonovich integrable on* $\Gamma$ if there exists the limit in probability

$$\lim_{|\Pi|\downarrow 0} \sum_{i=0}^{r(\Pi)-1} \frac{1}{\mu_{\Gamma_1}(\Delta_i^{(\Pi)})}\left(\int_{\Delta_i^{(\Pi)}} X(s, \tau(s))\,\mathrm{d}\mu_{\Gamma_1}(s)\right) W(D_i^{(\Pi)}),$$

where $\Pi = \{s_0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{r(\Pi)} = s_1\}$, $\Delta_i^{(\Pi)} = (\sigma_i, \sigma_{i+1}]$, and $D_i^{(\Pi)}$ is the closed area bounded by $V_{(\sigma_i, \tau(\sigma_i))}$, $V_{(\sigma_{i+1}, \tau(\sigma_{i+1}))}$, $\Gamma$ and the axis. This limit will be denoted by $I_s^{\Gamma_1}(X)$ or $\int_{\Gamma} X \circ \partial_1 W$, with a sign $+$ or $-$, in agreement with the type of $\Gamma$, sign plus if $\Gamma$ is of type I or II, sign minus if $\Gamma$ is of type I′ or II′. The class of 1-Stratonovich integrable processes on $\Gamma$ will be denoted by Dom $I_s^{\Gamma_1}$.

**Definition 4.10.** Let $X \in \mathbb{L}_\Gamma^{1,2}$. It is said that $X$ has 1-trace on $\Gamma$ if there exists the limit in probability

$$\lim_{|\Pi|\downarrow 0} \sum_i \frac{1}{\mu_{\Gamma_1}(\Delta_i^{(\Pi)})} \int_{\Delta_i^{(\Pi)} \times D_i^{(\Pi)}} D_zX(s, \tau(s))\,\mathrm{d}\mu_{\Pi_1}(s)\,\mathrm{d}z.$$

This limit will be denoted by $T^{\Gamma_1}X$.

**Remark 4.11.** As in Section 4.1 in Definitions 4.9 and 4.10 it suffices to consider sequences of refinements of $[s_0, s_1]$.

**Proposition 4.12.** *Let $X \in \mathbb{L}_{\Gamma}^{1,2}$. Then $X \in \operatorname{Dom} I_s^{\Gamma_1}$ if and only if $X$ has 1-trace on $\Gamma$. In this case,*

$$I_s^{\Gamma_1}(X) = \delta^{\Gamma_1}(X) + T^{\Gamma_1}(X).$$

**Proof.** It is the same as Theorem 1.9 of Solé and Utzet (1990). □

The most important case is when $\Gamma$ is a segment horizontal or vertical. This case is studied now:

**Notations 4.13.** Let $X \in \mathbb{L}^{1,2}$. It is said that $X \in \mathbb{L}_H$ if $X \in \mathbb{L}_{H(s,t)}^{1,2}$ for any $(s, t) \in T$ and there exists a neighbourhood $V$ of the diagonal $\{(x, x), x \in [0, s]\}$, $V \subset T$ such that for a given version of $\{D_z X_{z'}, z, z' \in T\}$:

(i) If we fix $(u, v) \in T$ and put $V_u^+ = \{u' : (u, u') \in V, u' > u\}$, the function

$$\begin{aligned} V_u^+ &\to L^2(\Omega) \quad \text{is continuous,} \\ u' &\to D_{uv} X_{u't} \quad \text{uniformly in } (u, v, t). \end{aligned}$$

(ii) The function

$$\begin{aligned} V_u^- &\to L^2(\Omega) \quad \text{is continuous,} \\ u' &\to D_{uv} X_{u't} \quad \text{uniformly in } (u, v, t), \end{aligned}$$

where $V_u^- = \{u' : (u, u') \in V, u' < u\}$.

(iii) $$\sup_{(u,v) \in V, (u',t) \in V} E[(D_{uv} X_{u't})^2] < \infty.$$

Define

$$D_{uv} X(u^+, t) = \lim_{u' \downarrow u} D_{uv} X_{u't} \quad \text{in } L^2(\Omega),$$

$$D_{uv} X(u^-, t) = \lim_{u' \uparrow u} D_{uv} X_{u't} \quad \text{in } L^2(\Omega).$$

There is a version of the process $\{D_{uv} X(u^+, t), (u, v, t) \in [0, 1]^3\} \in L^2([0, 1]^3 \times \Omega)$. It is also written

$$\nabla_{uv}^{(1)} X_{ut} = \tfrac{1}{2}(D_{uv} X(u^+, t) + D_{uv} X(u^-, t)).$$

In a similar way the space $\mathbb{L}_V$ and $\nabla_{uv}^{(2)} X_{sv}$ are defined.

It must be noted that, because of the uniform continuity, if $X \in L_H \cap L_V$,

$$\lim_{\substack{u' \downarrow u \\ r' \to r}} E[(D_{uv} X_{u'r'} - D_{uv} X_{u^+ r})^2] = 0.$$

**Proposition 4.14.** *Let $X \in \mathbb{L}_H$. Then $X$ has 1-trace on $H(s, t)$ and*

$$T_{H(s,t)}^1(X) = \frac{1}{2} \int_{R_{st}} (D_{uv} X(u^+, t) + D_{uv} X(u^-, t)) \, du \, dv.$$

**Proof.** The proof is very similar to the Proposition 4.7 and therefore omitted; see also Nualart and Pardoux (1988 Theorem 7.3). □

**Proposition 4.15.** *Let* $Y \in \mathbb{L}^{2,2}$, $\sup_z E[Y_z^2] < \infty$, $\sup_{z,z'} E[(D_z Y_{z'})^2] < \infty$ *and* $\sup_{z,z',z''} E[(D_{zz'}^2 Y_{z''})^2] < \infty$. *Define*

$$X_z = \int_{R_z} Y_\xi \, \mathrm{d}W_\xi.$$

*Then X has* 1*-trace on* $H(s,t)$ *and*

$$T^1_{H(s,t)}X = \frac{1}{2}\int_{R_{st}} Y_{uv}\,\mathrm{d}u\,\mathrm{d}v + \int_{R_{st}} \left(\int_{R_{ut}} D_{uv}Y_z\,\mathrm{d}W_z\right)\mathrm{d}u\,\mathrm{d}v. \qquad \square$$

*4.3. J-Stratonovich integral*

Let $\Pi = \{(s_i, t_j), 0 = s_0 < s_1 < \cdots < s_{n+1} = 1, 0 = t_0 < t_1 < \cdots < t_{m+1} = 1\}$ be a partition of $T$. Put

$$\delta_{ij}^\Pi = (0, s_i] \times (t_j, t_{j+1}] \quad \text{and} \quad \varepsilon_{ij}^\Pi = (s_i, s_{i+1}] \times (0, t_j].$$

Where there is no confusion, the superindex $\Pi$ will be omitted.

**Definition 4.16.** Let $X = \{X_z, z \in T\}$ be a measurable process such that $\int_T X_z^2\,\mathrm{d}z < \infty$, a.s. It is said that $X$ is *J-Stratonovich integrable* if there exists the limit in probability

$$I_J^s(X) = \lim_{|\Pi|\downarrow 0} \sum_{i,j} \frac{1}{|\Delta_{ij}|}\left(\int_{\Delta_{ij}} X_z\,\mathrm{d}z\right) W(\delta_{ij})\,W(\varepsilon_{ij}).$$

The set of $J$-Stratonovich integrable processes is denoted by Dom $I_J^s$. We also write

$$I_J^s(X) = \int_T X \circ \mathrm{d}J.$$

**Remark 4.17.** Suppose $X \in \mathbb{L}^{1,2}$. By Proposition 3.4 of Jolis and Sanz (1990); see also Lemma 1.2 of Nualart and Zakai (1988),

$$\begin{aligned}
&\sum_{i,j} \frac{1}{|\Delta_{ij}|}\left(\int_{\Delta_{ij}} X_z\,\mathrm{d}z\right) W(\delta_{ij})\,W(\varepsilon_{ij}) \\
&\quad = \delta^2\left(\sum_{i,j} \frac{1}{|\Delta_{ij}|}\left(\int_{\Delta_{ij}} X_z\,\mathrm{d}z\right)\mathbf{1}_{\delta_{ij}\otimes\varepsilon_{ij}}\right) \\
&\qquad + \delta\left(\sum_{i,j} \frac{1}{|\Delta_{ij}|}\left(\int_{\Delta_{ij}\times\varepsilon_{ij}} D_z X_{z'}\,\mathrm{d}z\,\mathrm{d}z'\right)\mathbf{1}_{\delta_{ij}}\right) \\
&\qquad + \delta\left(\sum_{i,j} \frac{1}{|\Delta_{ij}|}\left(\int_{\Delta_{ij}\times\delta_{ij}} D_z X_{z'}\,\mathrm{d}z\,\mathrm{d}z'\right)\mathbf{1}_{\varepsilon_{ij}}\right) \\
&\qquad + \sum_{i,j} \frac{1}{|\Delta_{ij}|}\int_{\Delta_{ij}\times\delta_{ij}\times\varepsilon_{ij}} D_{z,z'}X_\xi\,\mathrm{d}\xi\,\mathrm{d}z\,\mathrm{d}z'.
\end{aligned}$$

**Notations 4.18.** Let $\mathbb{L}_c^2$ be the set of processes $X \in \mathbb{L}^{2,2}$ such that there is a version of $\{D^2_{zz'}X_{z''}, (z, z', z'') \in T^3, z \curlywedge z'\}$ such that if $z=(s,t)$, $z'=(s',t')$ are fixed, the following conditions are satisfied:

(i) If $S^{++}_{zz'}$ denotes the set $\{z''=(s'',t''): s''>s', t''>t\}$, then the application

$$S^{++}_{zz'} \to L^2(\Omega), \qquad z'' \to D^2_{zz'}X_{z''},$$

is continuous, uniformly in $z, z'$.

(ii) Analogously $S^{+-}_{zz'}, S^{--}_{zz'}, S^{-+}_{zz'}$ are defined, and the maps $z'' \to D_{zz'}X_{z''}$ are continuous uniformly in $z, z'$.

$$\text{(iii)} \quad \sup_{z,z',z''} E[(D^2_{zz'}X_{z''})^2] < \infty.$$

Then we can define for $z_1 \curlywedge z_2$,

$$D^2_{z_1z_2}X_{z_1 \vee z_2^{++}} = L^2(\Omega) - \lim_{\substack{(s,t)\in S_{(s_1,t_1)(s_2,t_2)}^{++} \\ s\downarrow s_2, t\downarrow t_1}} D^2_{(s_1,t_1),(s_2,t_2)}X_{st}$$

and similarly, $D^2_{z_1z_2}X_{z_1\vee z_2^{-+}}$, $D^2_{z_1z_2}X_{z_1\vee z_2^{+-}}$ and $D^2_{z_1z_2}X_{z_1\vee z_2^{--}}$ are defined, and finally

$$\nabla^2_{z_1z_2}X_{z_1\vee z_2} = \tfrac{1}{4}(D^2_{z_1z_2}X_{z_1\vee z_2^{++}} + D^2_{z_1z_2}X_{z_1\vee z_2^{+-}} + D^2_{z_1z_2}X_{z_1\vee z_2^{-+}} + D^2_{z_1z_2}X_{z_1\vee z_2^{--}})$$

is defined.

The space $\mathbb{L}_{c'}^2$ is defined as the class of processes $X \in \mathbb{L}^{2,2}$ with a version of $\{D^2_{zz'}X_{z''}, (z,z',z'')\in T^3, z \curlywedge z' \text{ or } z<z'\}$ that verifies the same conditions as before. In the same way we can define $D^2_{z_1z_2}X(s_2^+, t_1^+)$, where $z_1=(s_1,t_1)$, $z_2=(s_2,t_2)$, and $\nabla^2_{z_1z_2}X_{z_1\vee z_2}$.

**Lemma 4.19.** *Let $X\in\mathbb{L}^{2,2}$ such that $X\in\mathbb{L}_H$ and for any $\xi\in T$, $D_\xi X\in\mathbb{L}_H$, and* $\sup_{\xi,\xi',z} E[(D^2_{\xi\xi'}X_z)^2]<\infty$. *Then $D_{\alpha\beta}X(\alpha^+, v)\in\mathbb{D}^{1,2}$ and*

$$D_{\alpha\beta}D_\xi X(\alpha^+, v) = D_\xi D_{\alpha\beta}X(\alpha^+, v), \quad \xi\text{-a.e.}$$

**Proof.**

$$D_{\alpha\beta}D_\xi X(\alpha^+, v) = L^2(\Omega) - \lim_n D_{\alpha\beta}D_\xi X(\alpha+1/n, v)$$

$$= \lim_n D_\xi D_{\alpha\beta}X(\alpha+1/n, v). \tag{4.2}$$

Therefore, for every $\xi\in T$, the sequence $\{D_\xi D_{\alpha\beta}X(\alpha+1/n, v), n\geq 1\}$ is a Cauchy sequence in $L^2(\Omega)$, and by dominated convergence $\{D.D_{\alpha\beta}X(\alpha+1/n, v), n\geq 1\}$ is Cauchy in $L^2(T\times\Omega)$ and since $D$ is a closed operator, $D_{\alpha\beta}X(\alpha^+, v)\in\mathbb{D}^{1,2}$ and

$$D.D_{\alpha\beta}X(\alpha^+, v) = L^2(T\times\Omega) - \lim_n D.D_{\alpha\beta}X(\alpha+1/n, v).$$

But the convergence in (4.1) implies that this limit is $D_{\alpha\beta}D.X(\alpha^+, v)$. And the lemma follows. □

A similar lemma for $D_{\alpha\beta}X(\alpha^-, v)$ and the vertical lines can be proved.

The following lemma will be needed in the next section.

**Lemma 4.20.** *Let* $X \in \mathbb{L}_V \cap \mathbb{L}^2_{c'}$, *such that for any* $\xi \in T$, $D_\xi X \in \mathbb{L}_V$. *Then a.e.* $(x, y, \alpha, \beta) \in T^2$, *with* $(x, y) < (\alpha, \beta)$ *or* $(x, y) \curlywedge (\alpha, \beta)$,

$$D^2_{(x,y),(\alpha,\beta)} X(\alpha^+, y^+) = L^2(\Omega) - \lim_{\gamma \downarrow \alpha} D_{\alpha\beta} D_{xy} X(\gamma, y^+).$$

**Proof.** By the Lemma 4.19, $D_{xy}X(\gamma, y^+) \in \mathbb{D}^{1,2}$, and then

$$\lim_{\gamma \downarrow \alpha} D_{\alpha\beta} D_{xy} X(\gamma, y^+) = \lim_{\gamma \downarrow \alpha} \lim_{\gamma' \downarrow y} D_{\alpha\beta} D_{xy} X_{\gamma y'}$$

and the lemma follows by the iterated limit theorem in $L^2(\Omega)$. $\square$

**Proposition 4.21.** *Let* $X \in \mathbb{L}_H \cap \mathbb{L}_V \cap \mathbb{L}^2_{c'}$ *such that for any* $\xi \in T$, $D_\xi X \in \mathbb{L}_H \cap \mathbb{L}_V$. *Then* $X \in \operatorname{Dom} I^s_J$ *and*

$$\begin{aligned}\int_T X \circ dJ = \int_T X \,\mathrm{d}J &+ \int_T \left( \int_T \nabla^{(1)}_{z'} X_{z \vee z'} \Psi(z, z')\,\mathrm{d}z' \right) \mathrm{d}W_z \\ &+ \int_T \left( \int_T \nabla^{(2)}_{z'} X_{z \vee z'} \Psi(z', z)\,\mathrm{d}z' \right) \mathrm{d}W_z \\ &+ \int_{T^2} \nabla^2_{zz'} X_{z \vee z'} \Psi(z, z')\,\mathrm{d}z\,\mathrm{d}z'.\end{aligned}$$

**Proof.** (1) To study the convergence

$$\sum_{i,j} \frac{1}{|\Delta_{ij}|} \left( \int_{\Delta_{ij}} X(\xi)\,\mathrm{d}\xi \right) \mathbf{1}_{\delta_{ij} \otimes \varepsilon_{ij}}(z, z') \xrightarrow{\mathbb{L}^{1,2}_{(2)}} X(z \vee z') \Psi(z, z')$$

consider the linear operators associated to the partitions $\{\Pi(n), n \in N\}$,

$$P_n X = \sum_{i,j} \frac{1}{|\Delta_{ij}|} \left( \int_{\Delta_{ij}} X(\xi)\,\mathrm{d}\xi \right) \mathbf{1}_{\delta_{ij} \otimes \varepsilon_{ij}}$$

and $PX = X(z \vee z')\Psi(z, z')$, and note that

$$\|P_n X\|_{L^2(T^2 \times \Omega)} \leqslant \|X\|_{L^2(T \times \Omega)}.$$

It is easy to see that if $X$ is continuous in $L^2$,

$$P_n X \to PX \quad \text{in } L^2(T^2 \times \Omega)$$

and the fact that the $L^2$-continuous processes are dense in $L^2(T \times \Omega)$, implies that the convergence $P_n X \to PX$ is true for any $X \in L^2(T \times \Omega)$. The convergence of first and second derivatives is proved in the same way and by applying the dominated convergence theorem.

(2) The convergence of other terms is proved using the Remark 4.17 and the same method as in Theorem 4.7. $\square$

The formula proved in Proposition 4.21 gives the relationship between the Stratonovich and the Skorohod integrals with respect to the process $J$. This process corresponds to the 'product measure' $\partial_1 W \partial_2 W$ (see Cairoli and Walsh, 1975). Therefore the appearance of the horizontal and vertical traces is not surprising.

*4.4. The Green–Stratonovich formula*

As in the Skorohod case, we need the notion of mixed Stratonovich integral. Let $X \in L^2(T\times\Omega)$ be a process such that fixed $(s,t)$, for any (a.e.) $r\in[0,t]$, $X\in \operatorname{Dom}\delta_S^{H(s,r)}$, and $\int_{H(s,r)} X\circ\partial_1 W$ is integrable in $r$. Then we define

$$\iint_{R_{st}} X\circ\partial_1 W\,\mathrm{d}r \overset{\text{def}}{=} \int_0^t \left(\int_{H(s,r)} X\circ\partial_1 W\right)\mathrm{d}r.$$

**Proposition 4.22.** *Let $X\in\mathbb{L}_H$. Then the mixed Stratonovich integral of $X$ exists on $R_{st}$ and it is equal to*

$$\iint_{R_{st}} X\,\partial_1 W\,\mathrm{d}r$$
$$+\frac{1}{2}\int_T\left[\int_0^t (D_{uv}X(u^+,r)+D_{uv}X(u^-,r))\mathbf{1}_{[v,1]}(r)\,\mathrm{d}r\right]\mathbf{1}_{[0,s]}(u)\,\mathrm{d}u\,\mathrm{d}v. \tag{4.3}$$

**Proof.** The proof is similar to the Proposition 4.4 and Theorem 4.7, and omitted. □

**Remark 4.23.** If $(s,t)=(1,1)$ the last term of (4.3) is equal to

$$\frac{1}{2}\int_T\left(\int_v^1 (D_{uv}X(u^+,r)+D_{uv}X(u^-,r))\,\mathrm{d}r\right)\mathrm{d}u\,\mathrm{d}v.$$

**Definition 4.24.** Given two measurable processes $f=\{f(z), z\in T\}$ and $g=\{g(z), z\in T\}$ such that for any $(s,r)\in T$, $f\in\operatorname{Dom}\delta_s^{V(s,r)}$ and $\int_0^1|g(s,v)|\,\mathrm{d}v<\infty$, a.e., consider the process

$$M_{sr}=M_{s0}+\int_{V_{sr}} f\circ\partial_2 W+\int_{V_{sr}} g\,\mathrm{d}v. \tag{4.4}$$

Then it is said that $M$ has 2-Stratonovich stochastic partial derivatives $f$ and $g$ in $T$.

We should remark that if $f\in\mathbb{L}_V^2$ the expression (4.4) may be written as

$$M_{sr}=M_{s0}+\int_{V_{sr}} f\,\partial_2 W+\int_{V_{sr}} h\,\mathrm{d}v,$$

where

$$h(s,v)=\frac{1}{2}\int_0^s (D_{uv}f(s,v^+)+D_{uv}f(s,v^-))\,\mathrm{d}u+g(s,v).$$

**Theorem 4.25** (Green–Stratonovich formula). *Let $M \in \mathbb{L}_a \cap \mathbb{L}_H$ a process with 2-Stratonovich stochastic partial derivatives $f$ and $g$ such that $f \in \mathbb{L}_H \cap \mathbb{L}_V \cap \mathbb{L}^2_{c'}$, for any $\xi \in T$, $D_\xi f \in \mathbb{L}_H \cap \mathbb{L}_V$, and $f$ is continuous in $L^2$ and $g \in \mathbb{L}_H$. Then for any rectangle $A \subset T$,*

$$\int_{\partial A} M \circ \partial_1 W = \int_A M \circ \mathrm{d}W + \int_A f \circ \mathrm{d}J + \iint_A g \circ \partial_1 W \,\mathrm{d}v + \frac{1}{4}\int_A f \,\mathrm{d}z. \tag{4.5}$$

**Proof.** For the same reasons as in the Skorohod case, it suffices to prove the theorem when $A$ is a rectangle $R_{st}$, and it can be assumed that $A = T$. We are going to check that when all integrals are written as Skorohod integrals plus their corresponding traces, the equality (4.5) is true. Assume $M$ zero on the axes.

Developing the expression (4.4) for $(s, t) = (\gamma, 1)$ we obtain:

$$M_{\gamma 1} = \int_T \mathbf{1}_{[0,\gamma]}(x) f(\gamma, y)\,\mathrm{d}W_{xy} + \frac{1}{2}\int_{R_{\gamma 1}} (D_{xy} f(\gamma, y^+) + D_{xy} f(\gamma, y^-))\,\mathrm{d}x\,\mathrm{d}y + \int_{V_{\gamma 1}} g(\gamma, y)\,\mathrm{d}y,$$

and

$$D_{\alpha\beta} M_{\gamma 1} = \mathbf{1}_{[0,\gamma]}(\alpha) f(\gamma, \beta) + \int_T \mathbf{1}_{[0,\gamma]}(x) D_{\alpha,\beta} f(\gamma, y)\,\mathrm{d}W_{xy} + \frac{1}{2}\int_{R_{\gamma 1}} (D_{\alpha\beta} D_{xy} f(\gamma, y^+) + D_{\alpha\beta} D_{xy} f(\gamma, y^-))\,\mathrm{d}x\,\mathrm{d}y + \int_{V_{\gamma 1}} D_{\alpha\beta} g(\gamma, y)\,\mathrm{d}y.$$

Therefore,

$$D_{\alpha\beta} M(\alpha^+, 1) = f(\alpha, \beta) + \int_T \mathbf{1}_{[0,\alpha]}(x) D_{\alpha\beta} f(\alpha^+, y)\,\mathrm{d}W_{xy} + \frac{1}{2}\int_{R_{\alpha 1}} (D^2_{(x,y),(\alpha,\beta)} f(\alpha^+, y^+) + D^2_{(x,y),(\alpha,\beta)} f(\alpha^+, y^-)\,\mathrm{d}x\,\mathrm{d}y + \int_{V_{\alpha 1}} D_{\alpha\beta} g(\alpha^+, y)\,\mathrm{d}y. \tag{4.6}$$

The proof of this is as follows: the first and fourth terms on the right of (4.6) are clear. For the second one, the Lemma 4.19 and dominated convergence are used.

For the third one, the Lemma 4.20 is applied. Note that, by the hypotheses, all terms in this equality are Lebesgue integrable in $(\alpha, \beta)$ except the Skorohod integral, and then, this term is also integrable.

The proof is completed by a straightforward calculus. $\square$

**Example 4.26.** Consider $M = W$. Then $f = 1$, and $g = 0$. The right hand of (4.5) is

$$\int_T W\,\mathrm{d}W + J_{11} + \tfrac{1}{2}$$

and $J_{11} = \frac{1}{2}W_{11}^2 - \int_T W\,\mathrm{d}W - \frac{1}{2}$, and therefore the right hand is $\frac{1}{2}W_{11}^2$.

The left hand of (4.5) is

$$\int_{\partial T} W \circ \partial_1 W = \int_{\partial T} W\,\partial_1 W + \tfrac{1}{2},$$

and by Itô's formula this is exactly $\frac{1}{2}W_{11}^2$.

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